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On the finite-size scaling of clusters in compact directed percolation

Michael J Kearney[1] **and Richard J Martin**[2]

[1] Advanced Technology Institute, School of Electronics and Physical Sciences, University of Surrey, Guildford, Surrey, GU2 7XH, UK
[2] Department of Mathematics and Centre for Nonlinear Dynamics and its Applications, University College London, Gower Street, London, WC1E 6BT, UK

E-mail: m.j.kearney@surrey.ac.uk

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Abstract

The exact finite-size scaling properties of clusters in compact directed percolation on a square lattice are derived. The results are implicit in previous work on the enumeration of staircase polygons, but their explicit form has not been presented as such before. The analysis provides important insights into the nature of this type of percolation transition.

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A key quantity of interest for percolation models is the probability, $P(p, s)$, of generating a cluster of a given size s [1, 2]. Scaling arguments suggest that $P(p, s)$ has the following form for $p \to p_c^-$ and $s \to \infty$

$$P(p, s) \sim s^{-\tau} f((p_c - p)^{1/\sigma} s) \qquad (1)$$

where p_c is the critical percolation probability and τ and σ are exponents characteristic of a particular universality class. However, very few exact results are known. The purpose of the present work is to demonstrate, by adapting results relating to the enumeration of staircase polygons, that one can *prove* (1) holds for compact directed percolation (CDP) on a square lattice, finding the exact exponent values and the exact finite-size scaling function, $f(t)$, in the process. Although relatively straightforward to deduce from what is already known, these results do not appear to have been written down explicitly before. As such, they provide important insights into the nature of the percolation transition in CDP.

A typical CDP cluster is shown in figure 1. Such clusters are grown (one diagonal row at a time) according to the rules set out in [3, 4] and are compact. The polygon (on the dual lattice) that bounds the cluster as tightly as possible is a staircase polygon [5–7]. All cluster-related

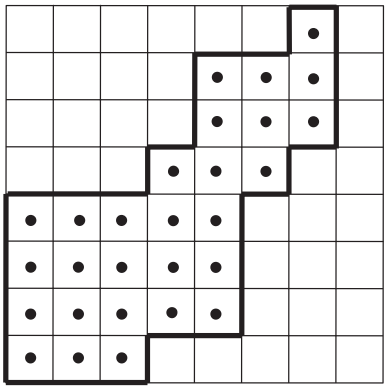

Figure 1. A typical CDP cluster (circles) grown (diagonally) from bottom left to top right in 13 time steps, together with its associated staircase polygon (solid line). The perimeter length is $\ell = 30$, and the cluster has $s = 28$ occupied sites. The activity of this particular polygon is $y^{30}z^{28}$; the probabilistic weight of the cluster is $p^{13}q^{15} = p^{-2}(pq)^{\ell/2}$.

quantities of interest can be derived from the generating function $G(y, z) \equiv \sum C_{\ell s} y^\ell z^s$, where $C_{\ell s}$ is the number of staircase polygons of perimeter length ℓ and area (or size) s. This is because in symmetric (i.e. single parameter) CDP a given cluster (assuming that the initial site is occupied with probability 1) has probability of occurrence $p^{-2}(pq)^{\ell/2}$, where $q = 1 - p$ [4]. Thus, letting $y = \sqrt{pq}$ weights every cluster correctly (apart from a trivial factor of p^{-2}). The discrete probability distribution, $P(p, s)$, is therefore defined by the appropriate term in the expansion of

$$p^{-2}G(\sqrt{pq}, z) \equiv \sum_s P(p, s)z^s. \tag{2}$$

We restrict our attention below to the regime $p \leqslant p_c = \frac{1}{2}$ where the clusters are finite with probability 1 [4]. The cluster size moments are then given by

$$S_k \equiv \langle s^k \rangle = p^{-2}\left(z\frac{\partial}{\partial z}\right)^k G(\sqrt{pq}, z)\bigg|_{z=1} \tag{3}$$

which follows directly from (2).

It is known from several different constructions that $G(y, z)$ satisfies a non-linear functional equation [6–9]

$$G(y, z) = y^4 z + 2y^2 z G(y, z) + G(y, z)G(y\sqrt{z}, z). \tag{4}$$

The generating function $G(y, z)$ has a tri-critical singularity at $y = y_c \equiv \frac{1}{2}$ and $z = z_c \equiv 1$ [6, 7], and it is this singularity that accounts for the phase transition in CDP. Using (3), and by repeatedly differentiating and rearranging (4), one can show that as $p \to p_c^-$ the moments diverge as

$$S_k \sim \frac{A_k}{(p_c - p)^{3k-1}}. \tag{5}$$

On the assumption that (1) is correct, it follows that $S_k \sim A_k(p_c - p)^{-\gamma_k}$ with $\gamma_k = (k - \tau + 1)/\sigma$. Only if $\tau = \frac{4}{3}$ and $\sigma = \frac{1}{3}$ is this consistent with (5) for all k. We shall prove below that these values are correct by establishing (1) directly.

The difficulty with analysing $P(p, s)$ is that the known representations for $G(y, z)$ [5–10] do not yield a natural expansion in the sense of (2). Nor is it possible to make direct use of the formal result,

$$P(p, s) \equiv p^{-2} \frac{1}{2\pi \mathrm{i}} \int_C G(\sqrt{pq}, z) \frac{\mathrm{d}z}{z^{s+1}} \tag{6}$$

where C is a contour that encloses the origin and no other singularity of $G(y, z)$. However, obtaining an asymptotic form for the generating function near the tri-critical point, and using this in (6), *is* sufficient to establish the behaviour of $P(p, s)$ as $p \to p_c^-$ and $s \to \infty$ [11]. Prellberg [7] has shown, starting from a q-series representation of $G(y, z)$, that in the limits $y \to y_c^-$ and $z \to 1^-$,

$$G(y, z) \sim \frac{1 - 2y^2}{2} + (1 - z)^{\theta} F\left(\frac{y_c - y}{(1 - z)^{\varphi}}\right) \tag{7}$$

where $\theta = \frac{1}{3}$ and $\varphi = \frac{2}{3}$, and

$$F(t) = \frac{1}{16} \frac{\mathrm{d}}{\mathrm{d}t} \ln \mathrm{Ai}(2^{8/3} t) \tag{8}$$

where Ai (t) is the Airy function. This result can also be obtained by assuming the form of (7) holds and expanding (4) about the tri-critical point using the method of dominant balance [12]. One finds that for $\theta = \frac{1}{3}$ and $\varphi = \frac{2}{3}$ the scaling function $F(t)$ obeys a non-linear (Riccati) equation,

$$F(t)^2 + \frac{1}{16} \frac{\mathrm{d}F(t)}{\mathrm{d}t} - t = 0$$

whose solution is given by (8). It is worth emphasizing that $F(t) < 0$ for $t \geqslant 0$ and it is useful to note that

$$F(0) = -\frac{3^{1/3}}{2^{4/3}} \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}.$$

For the special case $p = p_c$ (where $y = y_c$) it follows immediately from comparison of (2) and (7), and the use of (8), that

$$\sum_s P(p_c, s) z^s \equiv p_c^{-2} G\left(\tfrac{1}{2}, z\right) \sim 1 - 2\left(\frac{3}{2}\right)^{1/3} \frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} (1 - z)^{1/3}. \tag{9}$$

Note that when $z = 1$ the right-hand side of (9) is equal to 1, as required. Expanding and comparing coefficients gives for $s \to \infty$

$$P(p_c, s) \sim \left(\frac{2}{3}\right)^{2/3} \frac{1}{\Gamma\left(\frac{1}{3}\right)} s^{-4/3}. \tag{10}$$

This establishes that $\tau = \frac{4}{3}$ in (1), and also gives the value of $f(0)$.

Approximating $G(y, z)$ in (6) with its asymptotic form (7) raises subtle technical issues, a clear exposition of which may be found in [5, 11]. Below we simply highlight the main points (one can prove relatively easily that the results in [11] can be adapted to the present problem). For $p \to p_c^-$ and $s \to \infty$ we have

$$P(p, s) \sim p_c^{-2} \frac{1}{2\pi \mathrm{i}} \int_C (1 - z)^{1/3} F\left(\frac{(p_c - p)^2}{(1 - z)^{2/3}}\right) \frac{\mathrm{d}z}{z^{s+1}} \tag{11}$$

where we have used the fact that $y_c - y \sim (p_c - p)^2$. The validity of (11) can be justified using a theorem due to Darboux [11]. Further, the convergence properties of the Taylor series for $F(t)$ are such that one can expand and interchange the summation and integral to write

$$P(p,s) \sim 4\sum_{k=0}^{\infty}\frac{F^{(k)}(0)}{k!}(p_c-p)^{2k}\left[\frac{1}{2\pi\mathrm{i}}\int_C (1-z)^{(1-2k)/3}\frac{\mathrm{d}z}{z^{s+1}}\right]$$

where $F^{(k)}(0)$ is the kth derivative of $F(t)$ at zero argument. The contour integral is straightforward to evaluate by a residue calculation, with the result that for $s\to\infty$,

$$P(p,s) \sim 4s^{-4/3}\sum_{k=0}^{\infty}\frac{F^{(k)}(0)}{k!}\frac{(p_c-p)^{2k}s^{2k/3}}{\Gamma\left(\frac{2k-1}{3}\right)}. \tag{12}$$

This is exactly of the form (1) with exponents $\tau = \frac{4}{3}$ and $\sigma = \frac{1}{3}$. The scaling function itself is given by

$$f(t) = 4\sum_{k=0}^{\infty}\frac{F^{(k)}(0)}{k!}\frac{t^{2k/3}}{\Gamma\left(\frac{2k-1}{3}\right)}. \tag{13}$$

When $p = p_c$ only the $k = 0$ term contributes and one recovers (10). One can also use an integral representation of the reciprocal of the gamma function to write (13) in the equivalent form,

$$f(t) = \frac{2}{\pi\mathrm{i}}\int_{-\infty}^{(0+)} q^{1/3}\,\mathrm{e}^q F\left(\frac{t^{2/3}}{q^{2/3}}\right)\mathrm{d}q \tag{14}$$

where the contour is a Hankel contour. It would be desirable to have a 'simple' representation for the asymptotic form of $f(t)$ as $t\to\infty$. However, this remains an open problem.

As a corollary to the above, one can consider the cluster perimeter length probability distribution, $P(p,\ell)$, defined by

$$p^{-2}G(\sqrt{pq}\,y',1) \equiv \sum_{\ell} P(p,\ell)y'^{\ell} \tag{15}$$

where y' is a dummy variable introduced to keep track of the perimeter length. Setting $z = 1$ in (4) provides an algebraic equation for the perimeter generating function whose solution is

$$G(y,1) = \frac{1-2y^2-\sqrt{1-4y^2}}{2}. \tag{16}$$

From this result one can easily deduce the asymptotic behaviour of the perimeter moments as $p\to p_c^-$,

$$L_k = p^{-2}\left(y'\frac{\partial}{\partial y'}\right)^k G(\sqrt{pq}\,y',1)\bigg|_{y'=1} \sim \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(k-\frac{1}{2}\right)}{2^{k-1}(p_c-p)^{2k-1}}. \tag{17}$$

Further, expanding (16) gives, upon comparison with (15),

$$P(p,\ell) = \frac{1}{4p^2\sqrt{\pi}}\frac{\Gamma\left(\frac{\ell}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{\ell}{2}+1\right)}(4pq)^{\ell/2} \qquad \ell \text{ even.}$$

As $p\to p_c^-$ and $\ell\to\infty$ we therefore have the scaling form

$$P(p,\ell) \sim \frac{2^{3/2}}{\sqrt{\pi}}\ell^{-3/2}\exp\{-2\ell(p_c-p)^2\}. \tag{18}$$

In conclusion, by adapting results for staircase polygons it is possible to derive exact results for the probability of generating clusters of a given size (and perimeter) in compact directed percolation. The finite-size scaling functions and related exponents can all be identified, in full agreement with the standard scaling hypothesis. The analysis provides a clear yet deep insight into the nature of this type of percolation transition.

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